Paper Reference(s)

6668/01

Edexcel GCE

Further Pure Mathematics FP2

Advanced

Friday 19 June 2009 – Morning

Time: 1 hour 30 minutes

Materials required for examination	**Items included with question papers**
Mathematical Formulae (Orange)	Nil

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulas stored in them.

Instructions to Candidates

Write the name of the examining body (Edexcel), your centre number, candidate number, the unit title (Further Pure Mathematics FP2), the paper reference (6668), your surname, initials and signature.

Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
There are 8 questions in this question paper. The total mark for this paper is 75.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.

1. (*a*) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

(1)

(*b*) Hence show that $\sum_{r=1}^{n} \dfrac{4}{r(r+2)} = \dfrac{n(3n+5)}{(n+1)(n+2)}$.

(5)

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2. Solve the equation

$$z^3 = 4\sqrt{2} - 4\sqrt{2}\text{i},$$

giving your answers in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leq \pi$.

(6)

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3. Find the general solution of the differential equation

$$\sin x \frac{\text{d}y}{\text{d}x} - y\cos x = \sin 2x \sin x$$

giving your answer in the form $y = \text{f}(x)$.

(8)

4.

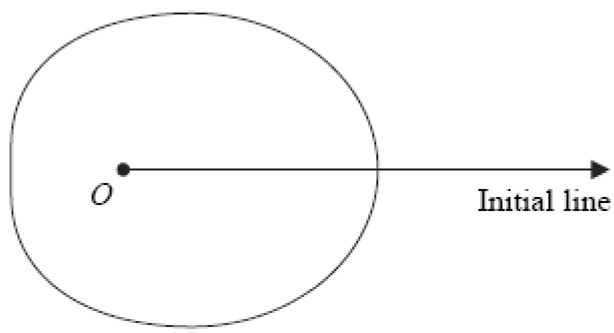

Figure 1

Figure 1 shows a sketch of the curve with polar equation

$$r = a + 3\cos\theta, \quad a > 0, \quad 0 \leq \theta < 2\pi.$$

The area enclosed by the curve is $\dfrac{107}{2}\pi$.

Find the value of a.

(8)

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5. $$y = \sec^2 x$$

(*a*) Show that $\dfrac{d^2 y}{dx^2} = 6\sec^4 x - 4\sec^2 x$.

(4)

(*b*) Find a Taylor series expansion of $\sec^2 x$ in ascending powers of $\left(x - \dfrac{\pi}{4}\right)$, up to and including the term in $\left(x - \dfrac{\pi}{4}\right)^3$.

(6)

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6. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + \mathrm{i}}, \quad z \neq -\mathrm{i}.$$

The circle with equation $|z| = 3$ is mapped by T onto the curve C.

(*a*) Show that C is a circle and find its centre and radius.

(8)

The region $|z| < 3$ in the z-plane is mapped by T onto the region R in the w-plane.

(*b*) Shade the region R on an Argand diagram.

(2)

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7. (*a*) Sketch the graph of $y = |x^2 - a^2|$, where $a > 1$, showing the coordinates of the points where the graph meets the axes.

(2)

(*b*) Solve $|x^2 - a^2| = a^2 - x$, $a > 1$.

(6)

(*c*) Find the set of values of x for which $|x^2 - a^2| > a^2 - x$, $a > 1$.

(4)

8.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}.$$

Given that $x = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$,

(*a*) find x in terms of t.

(8)

The solution to part (*a*) is used to represent the motion of a particle P on the x-axis. At time t seconds, where $t > 0$, P is x metres from the origin O.

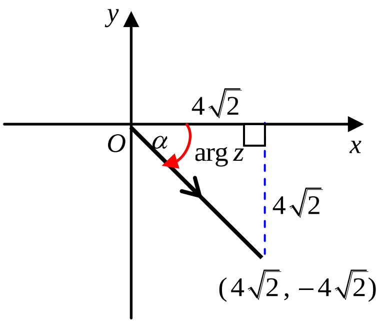
(*b*) Show that the maximum distance between O and P is $\frac{2\sqrt{3}}{9}$ m and justify that this distance is a maximum.

(7)

TOTAL FOR PAPER: 75 MARKS

END

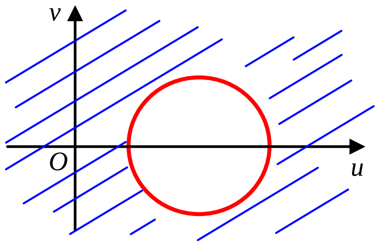
Question Number	Scheme	Marks
1. (a)	$\frac{1}{r(r+2)} = \underline{\frac{1}{2r} - \frac{1}{2(r+2)}}$	B1 (1)
(b)	$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{2}{r+2}\right)$	
	$= \left(\frac{2}{\underline{1}} - \frac{2}{3}\right) + \left(\frac{2}{\underline{2}} - \frac{2}{4}\right) + \ldots\ldots$	
	$\ldots\ldots\ldots\ldots + \left(\frac{2}{n-1} - \frac{2}{\underline{n+1}}\right) + \left(\frac{2}{n} - \frac{2}{\underline{n+2}}\right)$	M1
	$= \frac{2}{1} + \frac{2}{2} ; - \frac{2}{n+1} - \frac{2}{n+2}$	M1 A1
	$= 3 - \frac{2}{n+1} - \frac{2}{n+2}$	
	$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)}$	M1
	$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{(n+1)(n+2)}$	
	$= \frac{3n^2 + 5n}{(n+1)(n+2)}$	
	$= \frac{n(3n+5)}{(n+1)(n+2)}$	A1 cso (5)
		(6 marks)

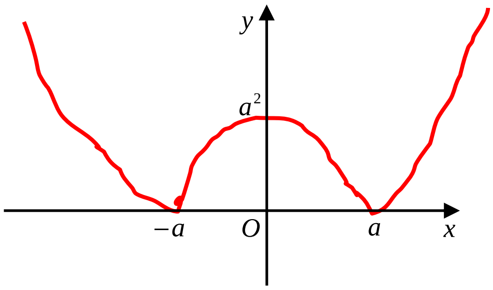
Question Number	Scheme	Marks
2. (a)	$z^3 = 4\sqrt{2} - 4\sqrt{2}\,\mathrm{i}\ ,\ \ -\pi < \theta \leq \pi$	
		
	$r = \sqrt{\left(4\sqrt{2}\right)^2 + \left(-4\sqrt{2}\right)^2} = \sqrt{32+32} = \sqrt{64} = 8$	
	$\theta = -\tan^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = -\frac{\pi}{4}$	M1
	$z^3 = 8\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$	
	So, $z = (8)^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{4}}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{4}}{3}\right)\right)$	M1
	$\Rightarrow z = 2\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$	A1
	Also, $z^3 = 8\left(\cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right)\right)$	
	or $z^3 = 8\left(\cos\left(-\frac{9\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{4}\right)\right)$	M1
	$\Rightarrow z = 2\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$	A1
	and $z = 2\left(\cos\left(\frac{-3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{-3\pi}{4}\right)\right)$	A1
		(6 marks)

Question Number	Scheme	Marks
3.	$\sin x \frac{dy}{dx} - y\cos x = \sin 2x \sin x$	
	$\frac{dy}{dx} - \frac{y\cos x}{\sin x} = \frac{\sin 2x \sin x}{\sin x}$	M1
	$\frac{dy}{dx} - \frac{y\cos x}{\sin x} = \sin 2x$	
	Integrating factor $= e^{\int -\frac{\cos x}{\sin x}dx} = e^{-\ln \sin x}$	M1 A1
	$= \frac{1}{\sin x}$	A1
	$\left(\frac{1}{\sin x}\right)\frac{dy}{dx} - \frac{y\cos x}{\sin^2 x} = \frac{\sin 2x}{\sin x}$	
	$\frac{d}{dx}\left(\frac{y}{\sin x}\right) = \sin 2x \times \frac{1}{\sin x}$	M1
	$\frac{d}{dx}\left(\frac{y}{\sin x}\right) = 2\cos x$	A1
	$\frac{y}{\sin x} = \int 2\cos x \, dx$	
	$\frac{y}{\sin x} = 2\sin x + K$	M1
	$y = 2\sin^2 x + K\sin x$	A1 cao
		(8 marks)

Question Number	Scheme	Marks
4.	$A = \frac{1}{2}\int_0^{2\pi} \left(a + 3\cos\theta\right)^2 \, d\theta$	B1
	$(a + 3\cos\theta)^2 = a^2 + 6a\cos\theta + 9\cos^2\theta$	
	$= \underline{a^2 + 6a\cos\theta + 9\left(\frac{1+\cos 2\theta}{2}\right)}$	M1 A1
	$A = \frac{1}{2}\int_0^{2\pi} \left(a^2 + 6a\cos\theta + \frac{9}{2} + \frac{9}{2}\cos 2\theta\right) d\theta$	
	$= \left(\frac{1}{2}\right)\left[a^2\theta + 6a\sin\theta + \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\right]_0^{2\pi}$	M1 A1 ft
	$= \frac{1}{2}\left[\left(2\pi a^2 + 0 + 9\pi + 0\right) - (0)\right]$	
	$= \pi a^2 + \frac{9\pi}{2}$	A1
	Hence, $\pi a^2 + \frac{9\pi}{2} = \frac{107}{2}\pi$	M1
	$a^2 + \frac{9}{2} = \frac{107}{2}$	
	$a^2 = 49$	
	As $a > 0$, $a = 7$	A1 cso
		(8 marks)

Question Number	Scheme	Marks
5.	$y = \sec^2 x = (\sec x)^2$	
(a)	$\frac{dy}{dx} = 2(\sec x)^1(\sec x \tan x) = 2\sec^2 x \tan x$	B1
	Apply product rule:	
	$\left\{\begin{matrix} u = 2\sec^2 x & v = \tan x \\ \frac{du}{dx} = 4\sec^2 x \tan x & \frac{dv}{dx} = \sec^2 x \end{matrix}\right\}$	
	$\frac{d^2y}{dx^2} = 4\sec^2 x \tan^2 x + 2\sec^4 x$	M1 A1
	$= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$	
	Hence, $\frac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$	A1
(b)	$y_{\frac{\pi}{4}} = \left(\sqrt{2}\right)^2 = \underline{2}$, $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = 2\left(\sqrt{2}\right)^2(1) = \underline{4}$	B1
	$\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 6\left(\sqrt{2}\right)^4 - 4\left(\sqrt{2}\right)^2 = 24 - 8 = 16$	M1
	$\frac{d^3y}{dx^3} = 24\sec^3 x(\sec x \tan x) - 8\sec x(\sec x \tan x)$	M1
	$= 24\sec^4 x \tan x - 8\sec^2 x \tan x$	
	$\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 24\left(\sqrt{2}\right)^4(1) - 8\left(\sqrt{2}\right)^2(1) = 96 - 16 = 80$	B1
	$\sec x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + \frac{16}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{80}{6}\left(x - \frac{\pi}{4}\right)^3 + \ldots$	M1 A1
	$\left\{\sec x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + 8\left(x - \frac{\pi}{4}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots\right\}$	
		(10 marks)

Question Number	Scheme	Marks
6.	$w = \frac{z}{z + \mathrm{i}}$, $z = -\mathrm{i}$	
	$w(z + \mathrm{i}) = z \Rightarrow wz + \mathrm{i}w = z \Rightarrow \mathrm{i}w = z - wz$	M1
(a)	$\Rightarrow \mathrm{i}w = z(1 - w) \Rightarrow z = \frac{\mathrm{i}w}{(1 - w)}$	A1
	$\|z\| = 3 \Rightarrow \left\|\frac{\mathrm{i}w}{1 - w}\right\| = 3$	M1
	$\left\{ \|\mathrm{i}w\| = 3\|1 - w\| \Rightarrow \|w\| = 3\|w - 1\| \Rightarrow \|w\|^2 = 9\|w - 1\|^2 \Rightarrow \|u + \mathrm{i}v\|^2 = 9\|u + \mathrm{i}v - 1\|^2 \right\}$	
	$\Rightarrow u^2 + v^2 = 9\left[(u-1)^2 + v^2\right]$	M1 A1
	$\left\{ \Rightarrow u^2 + v^2 = 9u^2 - 18u + 9 + 9v^2 \Rightarrow 0 = 8u^2 - 18u + 8v^2 + 9 \right\}$	
	$\Rightarrow 0 = u^2 - \frac{9}{4}u + v^2 + \frac{9}{8}$	M1
	$\Rightarrow \left(u - \frac{9}{8}\right)^2 + v^2 = \frac{9}{64}$	
	{Circle} centre $\left(\frac{9}{8}, 0\right)$, radius $\frac{3}{8}$	A1 A1 (8)
(b)		B1ft B1 (2)
		(10 marks)

Question Number	Scheme	Marks
7.	$y = \left\|x^2 - a^2\right\|,\ a > 1$	
(a)		B1 B1 (2)
(b)	$\left\|x^2 - a^2\right\| = a^2 - x\ ,\ a > 1$	
	$\{\|x\| > a\},\quad x^2 - a^2 = a^2 - x$	M1
	$\Rightarrow x^2 + x - 2a^2 = 0$	
	$\Rightarrow x = \dfrac{-1 \pm \sqrt{1 - 4(1)(-2a^2)}}{2}$	M1
	$\Rightarrow x = \dfrac{-1 \pm \sqrt{1 + 8a^2}}{2}$	A1
	$\{\|x\| < a\},\quad -x^2 + a^2 = a^2 - x$	M1
	$\{\Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0\}$	
	$\Rightarrow x = 0\,,1$	B1 A1 (6)
(c)	$\left\|x^2 - a^2\right\| > a^2 - x\ ,\ a > 1$	
	$x < \dfrac{-1 - \sqrt{1 + 8a^2}}{2}$ {or} $x > \dfrac{-1 + \sqrt{1 + 8a^2}}{2}$	B1 ft B1 ft
	{or} $0 < x < 1$	M1 A1 (4)
		(12 marks)

Question Number	Scheme	Marks
8.	$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$, $x = 0, \frac{dx}{dt} = 2$ at $t = 0.$	
(a)	AE, $m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0$	
	$\Rightarrow m = -3, -2.$	
	So, $x_{CF} = Ae^{-3t} + Be^{-2t}$	M1 A1
	$\left\{ x = ke^{-t} \Rightarrow \frac{dx}{dt} = -ke^{-t} \Rightarrow \frac{d^2x}{dt^2} = ke^{-t} \right\}$	
	$\Rightarrow ke^{-t} + 5(-ke^{-t}) + 6ke^{-t} = 2e^{-t} \Rightarrow 2ke^{-t} = 2e^{-t}$	M1
	$\Rightarrow k = 1$	A1
	$\{$So, $x_{PI} = e^{-t}\}$	
	So, $x = Ae^{-3t} + Be^{-2t} + e^{-t}$	M1
	$\frac{dx}{dt} = = -3Ae^{-3t} - 2Be^{-2t} - e^{-t}$	M1
	$t = 0, x = 0 \Rightarrow \quad 0 = A + B + 1$	
	$t = 0, \frac{dx}{dt} = 2 \Rightarrow \quad 2 = -3A - 2B - 1$	M1
	$\left\{ \begin{array}{l} 2A + 2B = -2 \\ -3A - 2B = 3 \end{array} \right\}$	
	$\Rightarrow A = -1, B = 0$	
	So, $x = -e^{-3t} + e^{-t}$	A1 cao (8)

Question Number	Scheme	Marks
8. (b)	$x = -e^{-3t} + e^{-t}$	
	$\frac{dx}{dt} = 3e^{-3t} - e^{-t} = 0$	M1
	$3 - e^{2t} = 0$	M1
	$\Rightarrow t = \frac{1}{2}\ln 3$	A1
	So, $x = -e^{-\frac{3}{2}\ln 3} + e^{-\frac{1}{2}\ln 3} = -e^{\ln 3^{-\frac{3}{2}}} + e^{\ln 3^{-\frac{1}{2}}}$	M1
	$x = -3^{-\frac{3}{2}} + 3^{-\frac{1}{2}}$	
	$= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} = \underline{\frac{2\sqrt{3}}{9}}$	A1
	$\frac{d^2x}{dt^2} = -9e^{-3t} + e^{-t}$	
	At $t = \frac{1}{2}\ln 3$, $\frac{d^2x}{dt^2} = -9e^{-\frac{3}{2}\ln 3} + e^{-\frac{1}{2}\ln 3}$	M1
	$= -9(3)^{-\frac{3}{2}} + 3^{-\frac{1}{2}} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}$	
	As $\frac{d^2x}{dt^2} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \left\{-\frac{2}{\sqrt{3}}\right\} < 0$ then x is maximum.	A1 (7)
		(15] marks)